Name: ___________________________________

Definitions

A number is **even** if it can be written in the form $2k$ for some integer k. A number is **odd** if it can be written in the form $2k+1$ for some integer k.

Given $n, m \in \mathbb{Z}$, we say that n **divides** m, written $n \mid m$, if there exists $k \in \mathbb{Z}$ such that $m = nk$. If $n \mid m$, we may also say m is **divisible** by n or n is a **factor** of m. If n **does not divide** m, we write $n \nmid m$.

Give a proof for each statement below.

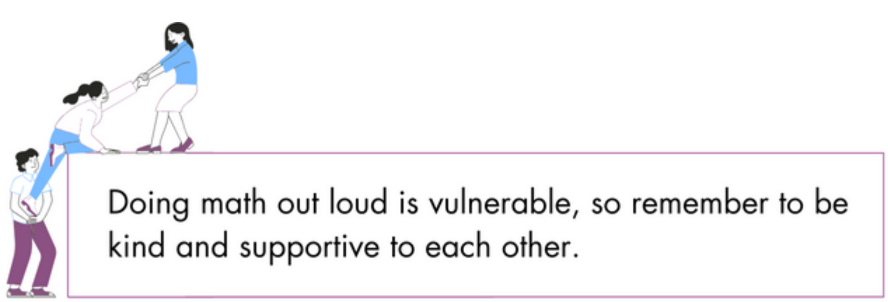
1. The sum of any three consecutive integers is divisible by three.

   Let n be an arbitrary integer. Then n, $n+1$, and $n+2$ are three consecutive integers. Their sum is $n + (n+1) + (n+2) = 3n + 3 = 3(n+1)$. Since $n+1$ is an integer, we see that the sum can be written as the product of 3 and an integer and thus the sum is divisible by 3.

   Slick solution: Let n be an arbitrary integer. Then $n-1$, n, and $n+1$ are three consecutive integers. Their sum is $(n-1) + n + (n+1) = 3n$. Since n is an integer, we see the sum is divisible by 3.

2. If $a \mid n$, then $a \mid mn$.

   Since $a \mid n$, we know $n = ak$ for some integer k. Therefore $mn = m(ak) = a(km)$. Since km is an integer, we see mn can be written as the product of a and some integer. We conclude that $a \mid mn$.



3. If $c, a, r \in \mathbb{R}$ such that $c \neq 0$ and $r \neq a/c$, then there exists a unique $x \in \mathbb{R}$ such that $(ax+1)/(cx) = r$.

   Existence: Since $r \neq a/c$, $y = \dfrac{1}{cr-a}$ is a real number. Then y satisfies the equation $(ax+1)/(cx) = r$, as shown below.

$$\begin{aligned} \frac{ay+1}{cy} &= \frac{a(\frac{1}{cr-a})+1}{c(\frac{1}{cr-a})} && (1)\\ &= \frac{a+cr-a}{c} && (2)\\ &= r && (3) \end{aligned}$$

   Note that equations (1) and (2) are well-defined because $c \neq 0$. We conclude that there exists a real number satisfying $(ax+1)/(cx) = r$.

   Uniqueness: Suppose real numbers y and z both satisfy the equation $(ax+1)/(cx) = r$. Then

$$\begin{aligned} \frac{ay+1}{cy} &= \frac{az+1}{cz} && (4)\\ acyz + cz &= acyz + cy && (5)\\ cz &= cy && (6)\\ z &= y && (7) \end{aligned}$$

   where the cancellation from line (3) to line (4) is valid because $c \neq 0$. Thus the real number satisfying $(ax+1)/(cx) = r$ is unique.

4. If ab divides n, then a divides n and b divides n.

   If ab divides n, we know there exists and integer k such that $(ab)k = n$. To see that a divides n, notice that $n = a(bk)$ where bk is an integer, so a is a factor of n. Similarly, to see that b divides n, notice that $n = b(ak)$ where ak is an integer, so b is a factor of n.

5. If n is odd, then 8 divides $n^2 - 1$.

Lemma 1. The product of consecutive integers is even.

Proof. Let k, $k+1$ be arbitrary consecutive integers.

Case 1: If k is even, then $k = 2j$ for some integer j. Then the product of k and $k+1$ is

$$\begin{aligned} k(k+1) &= (2j)(2j+1) \\ &= 4j^2 + 2j \\ &= 2(2j^2 + j) \end{aligned}$$

where $2j^2 + j$ is an integer. We see that the product is even.

Case 2: If k is odd, then $k = 2j+1$ for some integer j. Then the product of k and $k+1$ is

$$\begin{aligned} k(k+1) &= (2j+1)(2j+2) \\ &= 4j^2 + 6j + 2 \\ &= 2(2j^2 + 3j + 1) \end{aligned}$$

where $2j^2 + 3j + 1$ is an integer. We see that the product is even.

Proof of main statement. Since n is odd, there exists and integer k such that $n = 2k+1$. Thus

$$n^2 - 1 = (2k+1)^2 - 1 = 4k^2 + 4k = 4k(k+1).$$

Notice that $k(k+1)$ is the product of consecutive integers and must be even by Lemma 1, so there exists integer j such that $k(k+1) = 2j$. Finally

$$n^2 - 1 = 4 \cdot 2j = 8j$$

which proves that $n^2 - 1$ is divisible by 8.

6. If a divides b and b divides c, then $a \mid c$.

   If a divides b and b divides c, then there exist integers n and m such that $an = b$ and $bm = c$. Then $c = bm = (an)m = a(nm)$ where mn is an integer, which shows that $a \mid c$.

7. For all $n \in \mathbb{Z}$, $3n^2 + n + 14$ is even.

   (If n is even) If n is even then $n = 2k$ for some integer k. Then

$$\begin{aligned} 3n^2 + n + 14 &= 3(2k)^2 + 2k + 14 \\ &= 12k^2 + 2k + 14 \\ &= 2(6k^2 + k + 7) \end{aligned}$$

   where $6k^2 + k + 7$ is an integer. We conclude that $3n^2 + n + 14$ is even.

   (If n is odd) If n is odd then $n = 2k + 1$ for some integer k. Then

$$\begin{aligned} 3n^2 + n + 14 &= 3(2k + 1)^2 + (2k + 1) + 14 \\ &= 12k^2 + 14k + 8 \\ &= 2(6k^2 + 7k + 4) \end{aligned}$$

   where $6k^2 + 7k + 4$ is an integer. We conclude that $3n^2 + n + 14$ is even.